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Wave Function of a Free Electron in a Laser Plasma via Riemannian Geometry

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The wave function of a free electron in a laser plasma described via Riemannian geometry is derived by solving the Dirac equation in the associated curved space-time. If the laser field vanishes, the wave function naturally reduces to the case in flat space-time.

1. INTRODUCTION

The wave function and energy levels of a free electron are important objects. They are the basis for studying the free electron laser. In Zhu *et al.* (1995) we gave a Riemannian geometrical description for the laser plasma, but its correctness has not been proven. Because an electron or other charged particle is a microparticle, many phenomena concerning the electron have to be explained quantum mechanically or in terms of quantum fields in curved space-time. Therefore, the electron or other charged particle can be taken as a probe to detect the curvature of space-time. In this paper, the space-time structure with an optical metric is treated as a classical background, while the wave function of a free electron in the curved space-time is obtained by solving the Dirac equation. The obtained results can be used to test the geometry.

2. EQUATION

In order to obtain the wave function of a free electron in a laser plasma, we need to solve the Dirac equation in the curved space-time

$$(\gamma^\mu \nabla_\mu + mc/\hbar)\psi(x) = 0 \tag{1}$$

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where $\psi(x)$ is the wave function of a free electron, γ^μ is the Dirac matrix in the curved space-time, m is the electron mass, c is the velocity of light, $\hbar$ is the Planck constant, $\nabla_\mu \equiv \partial_\mu - \Gamma_\mu$ is the covariant derivative, and Γ_μ is the spinorial affine connection given by

$$\Gamma_\mu = -\frac{1}{4}\gamma^\lambda[\gamma_{\lambda,\mu} - \gamma_\nu \Gamma^\nu_{\lambda\mu}] + iqA_\mu \tag{2}$$

$\Gamma^\nu_{\lambda\mu}$ is the affine connection of the curved space-time, A_μ is the four-dimensional electromagnetic vector, q is the electron charge, and $\gamma_{\lambda,\mu}$ is the ordinary derivative of γ_λ.

Here the curved space-time is a moving inhomogeneous medium described with the optical metric given in Zhu *et al.* (1995). Specifically, it is the environment consisting of the self-consistent laser plasma in which the free electron moves.

When a y-polarized plane electromagnetic wave normally impinges onto a one-dimensional isothermally expanding plasma along the x direction, the optical metric of the self-consistent laser plasma system is given by

$$\bar{g}_{00} = -\frac{1}{1-N}, \qquad \bar{g}_{01} = \frac{c_s N_s}{c(1-N)}, \qquad \bar{g}_{11} = \bar{g}_{22} = \bar{g}_{33} = 1 \tag{3}$$

In order to remove the intersecting term, we make the coordinate transformation

$$x^0 = x^{0'} + F(x^1) \tag{4}$$

In the new coordinates $(x^{0'}, x^1, x^2, x^3)$ we have

$$\bar{g}_{00'} = \bar{g}_{00} = -\frac{1}{1-N} = -A, \qquad \bar{g}_{11'} = 1 + \frac{C_s^2 N_s^2}{C^2(1-N)} = B$$
$$\bar{g}_{22'} = \bar{g}_{33'} = 1 \tag{5}$$

and

$$F(x^1) = -\frac{\bar{g}_{01}}{\bar{g}_{00}} x^1 \tag{6}$$

For convenience in writing, we will omit the prime on the coordinate $x^{0'}$.

From equation (2), we obtain the nonzero components of the spinorial affine connection

$$\Gamma_0 = \frac{1}{4} A^{-1/2} B^{-1/2} A_{,1} \gamma^0 \gamma^1 \tag{7}$$

$$\Gamma_2 = iqA_2 \tag{8}$$

Here the four-dimensional electromagnetic vector A_μ is given in Shen and Zhu (1988) and the electron charge $q = -e$. Since the optical metric coefficients are only a function of the space coordinate x, the solution of equation (1) can be assumed to have the following form:

$$\psi(x, y, z, t) = \exp[i(p_2 y + p_3 z)/\hbar - iEt/\hbar]\phi(x) \tag{9}$$

Substituting equations (7)–(9) into equation (1), we have

$$[B^{-1/2}\gamma^1\partial_1 - iEA^{-1/2}\gamma^0/c\hbar + ip_2\gamma^2/\hbar + ip_3\gamma^3/\hbar$$
$$+ \frac{1}{4}A^{-1}B^{-1/2}A_{,1}\gamma^1 - ieA_2\gamma^2 + mc/\hbar]\phi(x) = 0 \tag{10}$$

Multiplying equation (10) by $B^{1/2}\gamma^1$ from the left yields

$$\partial_1\phi(x) = \Big[iEA^{-1/2}B^{1/2}\gamma^1\gamma^0/c\hbar$$
$$- iB^{1/2}\gamma^1(p_2\gamma^2 + p_3\gamma^3)/\hbar - \frac{1}{4}A^{-1}A_{,1}$$
$$+ ieA_2B^{1/2}\gamma^1\gamma^2 - mcB^{1/2}\gamma^1/\hbar\Big]\phi(x) \tag{11}$$

The Dirac matrix γ^μ is given by the following relations:

$$\gamma^0 = -\gamma_0 = -i\beta \tag{12}$$

$$\gamma^i = -\gamma_i = -i\beta\alpha_i \tag{13}$$

$$\alpha^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} \tag{14}$$

$$\beta = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \tag{15}$$

with

$$\beta^2 = 1 \tag{16}$$

$$\alpha_i\alpha_j = 2\delta_{ij} \tag{17}$$

$$\alpha_i\beta + \beta\alpha_i = 0 \tag{18}$$

where σ^i is the Pauli matrix and satisfies

$$\sigma_i\sigma_j = i\sigma_k \tag{19}$$

i, j, k is the cyclic permutation of 1, 2, 3. Equation (11) reduces to

$$\partial_1\phi(x) = \left[iEA^{-1/2}B^{1/2}\alpha^1/c\hbar + B^{1/2}(p_2\sigma^3 - p_3\sigma^2)/\hbar - \frac{1}{4}A^{-1}A_{,1} - eA_2B^{1/2}\sigma^3 + imcB^{1/2}\beta\alpha^1/\hbar\right]\phi(x) \tag{20}$$

Solving equation (20), we obtain

$$\phi(x) = A^{-1/4}\exp(iE\alpha^1X_1 + imc\beta\alpha^1X_0 + \sigma^3X_2 - \sigma^2X_3)\phi_c \tag{21}$$

Substituting equation (21) into equation (9), we get

$$\begin{aligned}\psi(x, y, z, t) &= A^{-1/4}\exp[i(p_2y + p_3z)/\hbar - iEt/\hbar \\ &\quad + iE\alpha^1X_1 + imc\beta\alpha^1X_0 \\ &\quad + \sigma^3X_2 - \sigma^2X_3]\phi_c \\ &= A^{-1/4}\exp[i(p_2y + p_3z)/\hbar - iEt/\hbar - i\hat{\eta}]\,\phi_c\end{aligned} \tag{22}$$

where ϕ_c is a constant spinor, and $\hat{\eta}$ is a 4×4 matrix, i.e.,

$$\begin{aligned}\hat{\eta} &= -E\alpha^1X_1 - mc\beta\alpha^1X_0 + i\sigma^3X_2 - i\sigma^2X_3 \\ &= \begin{pmatrix} iX_2 & -X_3 & 0 & -EX_1 + mcX_0 \\ X_3 & -iX_2 & -EX_1 + mcX_0 & 0 \\ 0 & -EX_1 + mcX_0 & iX_2 & -X_3 \\ -EX_1 + mcX_0 & 0 & X_3 & -iX_2 \end{pmatrix}\end{aligned} \tag{23}$$

with

$$X_0 = \int B^{1/2}/\hbar\, dx \tag{24}$$

$$X_1 = \int A^{-1/2}B^{1/2}/c\hbar\, dx \tag{25}$$

$$X_2 = \int B^{1/2}(p_2/\hbar - eA_2)\, dx \tag{26}$$

$$X_3 = \int B^{1/2} p_3/\hbar \, dx \tag{27}$$

The square of the 4×4 matrix $\hat{\eta}$ is the product of the function factor η^2 and the 4×4 unit matrix, i.e.,

$$\hat{\eta}^2 = \eta^2 I \tag{28}$$

with

$$\eta = (E^2 X_1^2 - m^2 c^2 X_0^2 - X_2^2 - X_3^2)^{1/2} \tag{29}$$

Hence the exponential function of $\hat{\eta}$ can be written as

$$\exp(-i\hat{\eta}) = \cos \eta - (i\hat{\eta} \sin \eta)/\eta \tag{30}$$

Then equation (22) can be rewritten as

$$\begin{aligned} \psi(x, y, z, t) &= A^{-1/4} \exp[i(p_2 y + p_3 z)/\hbar - iEt/\hbar] \\ &\quad \times [\cos \eta - (i\hat{\eta} \sin \eta)/\eta]\phi_c \end{aligned} \tag{31}$$

3. DETERMINATION OF THE CONSTANT SPINOR ϕ_c

As might be expected, when the external field vanishes, the wave function should reduce to the plane-wave solution of a relativistic free electron in the flat space-time (Schiff, 1968)

$$\psi_{p,E} = u(p) \exp[i(p \cdot r - Et)/\hbar] \tag{32}$$

with

$$u(p)^{(1)} = N \begin{pmatrix} 1 \\ 0 \\ cp_3(mc^2 + E_+) \\ c(p_1 + ip_2)/(mc^2 + E_+) \end{pmatrix} \tag{33}$$

$$u(p)^{(2)} = N \begin{pmatrix} 0 \\ 1 \\ c(p_1 - ip_2)/(mc^2 + E_+) \\ -cp_3/(mc^2 + E_+) \end{pmatrix} \tag{34}$$

$$u(p)^{(3)} = N \begin{pmatrix} cp_3/(-mc^2 + E_-) \\ c(p_1 + ip_2)/(-mc^2 + E_-) \\ 1 \\ 0 \end{pmatrix} \tag{35}$$

$$u(p)^{(4)} = N \begin{pmatrix} c(p_1 - ip_2)/(-mc^2 + E_-) \\ -cp_3/(-mc^2 + E_-) \\ 0 \\ 1 \end{pmatrix} \tag{36}$$

where the normalized constant N is equal to

$$N = [1 + c^2p^2/(mc^2 + |E|^2)]^{-1/2} \tag{37}$$

If the constant spinor ϕ_c is written as

$$\phi_c = \begin{pmatrix} a \\ b \\ d \\ f \end{pmatrix} \tag{38}$$

then its Hermitian conjugate is

$$\phi_c^+ = \tilde{\phi}_c^* = (a^*, b^*, d^*, f^*) \tag{39}$$

When the external field vanishes, i.e., $A_2 = 0$, we obtain

$$A = 1 \tag{40}$$

$$B = 1 \tag{41}$$

and from equations (24)–(27) we have

$$X_0 = x/\hbar \tag{42}$$

$$X_1 = x/c\hbar \tag{43}$$

$$X_2 = p_2x/\hbar \tag{44}$$

$$X_3 = p_3x/\hbar \tag{45}$$

Substituting equations (42)–(45) into equation (29) gives

$$\eta = p_1x/\hbar \tag{46}$$

From equations (23) and (30), we get

$$\hat{\eta} = \begin{pmatrix} ip_2 & -p_3 & 0 & (-E + mc^2)/c \\ p_3 & -ip_2 & (-E + mc^2)/c & 0 \\ 0 & (-E + mc^2)/c & ip_2 & -p_3 \\ (-E + mc^2)/c & 0 & p_3 & -ip_2 \end{pmatrix} \frac{x}{\hbar} \tag{47}$$

and

$$\begin{aligned} \exp(-i\hat{\eta}) &= \cos\eta - (i\hat{\eta}\sin\eta)/\eta \\ &= \left(\begin{matrix} \cos\left(\frac{p_1 x}{\hbar}\right) + \frac{p_2}{p_1}\sin\left(\frac{p_1 x}{\hbar}\right) & i\frac{p_3}{p_1}\sin\left(\frac{p_1 x}{\hbar}\right) \\ -i\frac{p_3}{p_1}\sin\left(\frac{p_1 x}{\hbar}\right) & \cos\left(\frac{p_1 x}{\hbar}\right) - \frac{p_2}{p_1}\sin\left(\frac{p_1 x}{\hbar}\right) \\ 0 & \frac{i(E - mc^2)}{cp_1}\sin\left(\frac{p_1 x}{\hbar}\right) \\ \frac{i(E - mc^2)}{cp_1}\sin\left(\frac{p_1 x}{\hbar}\right) & 0 \end{matrix}\right. \\ &\qquad \left.\begin{matrix} 0 & \frac{i(E - mc^2)}{cp_1}\sin\left(\frac{p_1 x}{\hbar}\right) \\ \frac{i(E - mc^2)}{cp_1}\sin\left(\frac{p_1 x}{\hbar}\right) & 0 \\ \cos\left(\frac{p_1 x}{\hbar}\right) + \frac{p_2}{p_1}\sin\left(\frac{p_1 x}{\hbar}\right) & i\frac{p_3}{p_1}\sin\left(\frac{p_1 x}{\hbar}\right) \\ -i\frac{p_3}{p_1}\sin\left(\frac{p_1 x}{\hbar}\right) & \cos\left(\frac{p_1 x}{\hbar}\right) - \frac{p_2}{p_1}\sin\left(\frac{p_1 x}{\hbar}\right) \end{matrix}\right) \end{aligned} \tag{48}$$

As the external field vanishes, the wave function [equation (31)] of a free electron in the laser plasma should be reduced to the wave function [equation (32)] of a relativistic free electron in the flat space-time, i.e.,

$$\begin{aligned} \psi(x, y, z, t) &= \exp[i(p_2 y + p_3 z)/\hbar - iEt/\hbar] \\ &\qquad \times [\cos\eta - (i\hat{\eta}\sin\eta)/\eta]\phi_c \\ &= u(p)\exp[i(p_1 x + p_2 y + p_3 z)/\hbar - iEt/\hbar] \end{aligned} \tag{49}$$

This gives

$$[\cos\eta - (i\hat{\eta}\sin\eta)/\eta]\phi_c = u(p)^{(j)}\exp(ip_1 x/\hbar) \qquad (j = 1, 2, 3, 4) \tag{50}$$

If we choose $j = 1$, equation (50) becomes

$$[\cos\eta - (i\hat{\eta}\sin\eta)/\eta]\begin{pmatrix} a \\ b \\ d \\ f \end{pmatrix}$$

$$= N\begin{pmatrix} 1 \\ 0 \\ cp_3(mc^2 + E_+) \\ c(p_1 + ip_2)/(mc^2 + E_+) \end{pmatrix}\exp(ip_1x/\hbar) \tag{51}$$

Then we have

$$a\left[\cos\left(\frac{p_1x}{\hbar}\right) + \frac{p_2}{p_1}\sin\left(\frac{p_1x}{\hbar}\right)\right] + ib\frac{p_3}{p_1}\sin\left(\frac{p_1x}{\hbar}\right)$$

$$+ if\frac{E + mc^2}{cp_1}\sin\left(\frac{p_1x}{\hbar}\right) = N\exp\left(\frac{ip_1x}{\hbar}\right) \tag{52}$$

$$ia\frac{p_3}{p_1}\sin\left(\frac{p_1x}{\hbar}\right) + b\left[\cos\left(\frac{p_1x}{\hbar}\right) - \frac{p_2}{p_1}\sin\left(\frac{p_1x}{\hbar}\right)\right]$$

$$+ d\frac{E + mc^2}{cp_1}\sin\left(\frac{p_1x}{\hbar}\right) = 0 \tag{53}$$

$$ib\frac{E - mc^2}{cp_1}\sin\left(\frac{p_1x}{\hbar}\right) + d\left[\cos\left(\frac{p_1x}{\hbar}\right) + \frac{p_2}{p_1}\sin\left(\frac{p_1x}{\hbar}\right)\right]$$

$$+ if\frac{p_3}{p_1}\sin\left(\frac{p_1x}{\hbar}\right) = Ncp_3(mc^2 + E_-)\exp\left(\frac{ip_1x}{\hbar}\right) \tag{54}$$

$$ia\frac{E - mc^2}{cp_1}\sin\left(\frac{p_1x}{\hbar}\right) - id\frac{p_3}{p_1}\sin\left(\frac{p_1x}{\hbar}\right)$$

$$+ f\left[\cos\left(\frac{p_1x}{\hbar}\right) - \frac{p_2}{p_1}\sin\left(\frac{p_1x}{\hbar}\right)\right]$$

$$= Nc(p_1 + ip_2)\exp\left(\frac{ip_1x}{\hbar}\right)/(mc^2 + E_+) \tag{55}$$

Comparing the coefficients of the same exponent terms on both sides of the equalities in (52)–(55), we obtain

$$a = N \tag{56}$$

$$b = 0 \tag{57}$$

$$d = Ncp_3(mc^2 + E_+) \tag{58}$$

$$f = Nc(p_1 + ip_2)/(mc^2 + E_+) \tag{59}$$

Then the first constant spinor is

$$\phi_c^{(1)} = \begin{pmatrix} a \\ b \\ d \\ f \end{pmatrix} = N\begin{pmatrix} 1 \\ 0 \\ cp_3(mc^2 + E_+) \\ c(p_1 + ip_2)/(mc^2 + E_+) \end{pmatrix} \tag{60}$$

If we choose $j = 2, 3, 4$, respectively, we can obtain the other constant spinors

$$\phi_c^{(2)} = N\begin{pmatrix} 0 \\ 1 \\ c(p_1 - ip_2)/(mc^2 + E_+) \\ -cp_3/(mc^2 + E_+) \end{pmatrix} \tag{61}$$

$$\phi_c^{(3)} = N\begin{pmatrix} cp_3/(-mc^2 + E_-) \\ c(p_1 + ip_2)/(-mc^2 + E_-) \\ 1 \\ 0 \end{pmatrix} \tag{62}$$

$$\phi_c^{(4)} = N\begin{pmatrix} c(p_1 - ip_2)/(-mc^2 + E_-) \\ -cp_3/(-mc^2 + E_-) \\ 0 \\ 1 \end{pmatrix} \tag{63}$$

The four wave functions of a free electron in a laser plasma can be written as

$$\psi^{(1)} = A^{-1/4} \exp[i(p_2 y + p_3 z)/\hbar - iE_+ t/\hbar] \times [\cos\eta - (i\hat{\eta}\sin\eta)/\eta]\phi_c^{(1)} \tag{64}$$

$$\psi^{(2)} = A^{-1/4} \exp[i(p_2 y + p_3 z)/\hbar - iE_+ t/\hbar] \times [\cos\eta - (i\hat{\eta}\sin\eta)/\eta]\phi_c^{(2)} \tag{65}$$

$$\psi^{(3)} = A^{-1/4} \exp[i(p_2 y + p_3 z)/\hbar - iE_- t/\hbar] \times [\cos\eta - (i\hat{\eta}\sin\eta)/\eta]\phi_c^{(3)} \tag{66}$$

$$\psi^{(4)} = A^{-1/4} \exp[i(p_2 y + p_3 z)/\hbar - iE_- t/\hbar] \times [\cos\eta - (i\hat{\eta}\sin\eta)/\eta]\phi_c^{(4)} \tag{67}$$

with

$$E_+ = (c^2 p^2 + m^2 c^4)^{1/2} \tag{68}$$

$$E_- = -E_+ = -(c^2 p^2 + m^2 c^4)^{1/2} \tag{69}$$

where $\phi_c^{(1)}$, $\phi_c^{(2)}$, $\phi_c^{(3)}$, $\phi_c^{(4)}$, $\hat{\eta}$, and η are given by equations (60)–(63), (23), and (29), respectively.

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